TWISTOR SPACES OF NON-FLAT BOCHNER-KÄHLER MANIFOLDS

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1. INTRODUCTION

It was shown by Kamishima in [4] that a Bochner-Kähler manifold of dimension $n > 1$ is locally isomorphic to one of the following model spaces:

1. a flat geometry $(\mathbb{C}^n \times (\mathrm{U}(n) \times \mathbb{R}_+), \mathbb{C}^n)$,
2. a non-flat geometry $(\mathrm{PU}(1, l) \times \mathrm{PU}(m + 1), \mathbf{H}^l \times \mathbf{P}^m)$.

This can be compared to the fact that a conformally flat manifold of dimension $n > 2$ is locally conformal to a region of the sphere of the same dimension. In twistor theory, it is well-known that an even dimensional conformally flat manifold has an integrable twistor space ([7], [6], [1], [3]). It is interesting, as an analogy, that a Bochner-Kähler manifold has integrable twistor spaces defined by O'Brian and Rawnsley in [7]. Since it was proved by calculating the Nijenhuis tensors, the proof does not give holomorphic coordinates. Hence it would be natural to ask to "give a system of local coordinates of the twistor spaces of the above model spaces". Furthermore, to study a generalization of the Penrose transform, it is necessary to "construct the moduli space of relative deformations of fibers", which we call the *complexification*.

In the present paper, we give answers to the above two questions.

Since the first space is flat, its twistor space as a Riemannian manifold has already an integrable complex structure. Hence the problems can be solved easily by restricting the Penrose diagram as a Riemannian manifold.

In the second case, we give local coordinates by constructing a local embedding to a product of complex projective spaces (Theorem 3.1 and Theorem 3.3). The complexification of $\mathbf{H}^l \times \mathbf{P}^m$ is $\mathbf{H}^l \times \bar{\mathbf{H}}^l \times \left(\mathbf{P}^m \times \bar{\mathbf{P}}^m \setminus F(1, m; m + 1) \right)$ (Corollary 4.3).

Let us explain briefly the contents of this paper. In §2, we review the definition of the twistor spaces of Hermitian manifolds. The answers in the flat case are explained shortly. In §3, we construct a system of local coordinates of the twistor spaces in the second case. In §4, we give

an explicit description of the complexification of the model space, by extending maps in the previous section. In §5, we give some examples.

2. The twistor spaces of Hermitian manifolds

In this section, we review the definition of the twistor spaces of a general Hermitian manifold M. Let n be the complex dimension of M and k an integer between 0 and n. The k-th twistor space $Z_k(M)$ of M is the total space of the Grassmannian bundle $G_k(T^{(1,0)}M)$ equipped with the almost complex structure defined as follows. By the Hermitian metric on M, we can define a connection on its frame bundle. Since $G_k(M)$ is an associated bundle of the frame bundle, the connection decomposes the real tangent bundle of $Z_k(M)$ as

$$TZ_k(M) = T_V Z_k(M) \oplus T_H Z_k(M).$$

Since its fibers have natural holomorphic structures, we have a complex structure on the first component. For $x \in M$ and $L \in G_k(T_x^{(1,0)}M)$, decompose the second component as

$$T_H Z_k(M)_{(x,L)} = T_x^{(1,0)}M = L \oplus L^\perp.$$

This decomposition is compatible with the original complex structure, since L is a complex subspace of $T_x^{(1,0)}M$. We define a new complex structure on $T_H Z_k(M)_{(x,L)}$ by replacing the complex structure on L with its conjugate structure.

We can write explicitly the space of horizontal $(1,0)$-forms, by using the Plücker coordinates with notation of multi-indices similar to that in [2] and [3]. Let $(e_1, \ldots, e_n)$ be a local orthonormal basis of the space of $(1,0)$-vectors of M. For $I_0 = (i_1 \ldots i_k)$ with $i_1 < \cdots < i_k$, let z^{I_0} be the coefficient of $e_{I_0} = e_{i_1} \wedge \cdots \wedge e_{i_k}$. A multi-index I is a sequence of elements of $\{1, \ldots, n\}$. The composition of the sequences I and J is written as IJ. We put relations between z^I's as:

$$z^{I_1 i i I_2} = -z^{I_1 I_2},$$
$$z^{I_1 i j I_2} = -z^{I_1 j i I_2}, \quad i \neq j.$$

Then, for any multi-index I, there is a unique subsequence I_0 of $(1, \ldots, n)$ such that $z^I = \pm z^{I_0}$. We denote $|I|$ to be the length of I_0. Then the space of horizontal $(1,0)$-forms of the twistor space is spanned by the following forms:

(2.1)
$$\sum_{j \notin I} z^{jI} \bar{e}^j, \quad |I| = k - 1,$$
$$\sum_{i \in I} z^{iI} e^i, \quad |I| = k + 1,$$

where $(e^1, \ldots, e^n)$ is the dual basis.

The integrability condition of the almost complex structure is given by O'Brian and Rawnsley in [7]. A Bochner-Kähler manifold, which we will study in the following sections, is an example of such manifolds.

Henceforth, we assume that the almost complex structure of $Z_k(M)$ is integrable.

We can define the complexification of an even dimensional conformally flat manifold by considering the relative deformations of fibers of its twistor space. Since Grassmannian manifolds are rigid, we can also define the *complexification* of M in a similar way.

For $(A, B) \in \mathrm{U}(k) \times \mathrm{U}(n - k)$, define

$$\rho(A, B) : \mathbb{C}^k \oplus \mathbb{C}^{n-k} \to \mathbb{C}^k \oplus \mathbb{C}^{n-k}$$
$$(v_1, v_2) \mapsto (A^* v_1, B v_2).$$

Then, for $x \in M$, the normal bundle N_x of the fiber $Z_k(M)_x$ is written as

$$N_x \simeq N = \mathrm{U}(n) \times_\rho (\mathbb{C}^k \oplus \mathbb{C}^{n-k}).$$

By the theorem of Bott-Borel-Weil-Kostant, we have:

$$H^i(Z_k(M)_x, \mathcal{O}(N_x)) = \begin{cases} \Lambda_x^{(1,0)} M \oplus T_x^{(1,0)} M, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

Hence, by the relative deformation theory of Kodaira ([5]), the moduli space

$$M_{\mathbb{C}} = \{Z \subset Z_k(M) \mid Z \text{ is a deformation of a fiber.}\}$$

is a $2n$-dimensional complex manifold, which includes M as the space of *real* fibers.

As an example, we describe the situation shortly when $M = \mathbb{C}^n$ with flat metric.

Let $X = \mathbb{C}^n \times \bar{\mathbb{C}}^n = H^0(G_{k,n}, \mathcal{O}(N))$. Let Y be the submanifold of $N \times X$ defined by

$$Y = \{(f(z), f) \mid f \in X, z \in G_{k,n}\}.$$

We have a canonical double fibration:

$$\begin{array}{ccc} & Y & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ N & & X. \end{array}$$

Then we can show easily that, for each $w \in N$, there is a unique element $v \in \mathbb{C}^n$ such that $w \in p_1 \circ p_2^{-1}(v, \bar{v})$. By the projection $w \mapsto v$, we can identify N with $Z_k(\mathbb{C}^n)$. The *complexification* of $\mathbb{C}^n$ is X, namely, for $x \in X$, the corresponding submanifold is written as $p_1 \circ p_2^{-1}(x)$.

Remark. Since the twistor space $Z(\mathbb{C}^n)$ as a Riemannian manifold satisfies the integrability condition, this diagram can be obtained by restricting the Penrose double fibration as a Riemannian manifold, which is described in detail by Murray in [6].

3. The model spaces of non-flat Bochner-Kähler manifolds and their twistor spaces

Let V be a complex vector space of dimension $l + 1$ having the indefinite metric:
$$\begin{pmatrix} 1 & 0 \\ 0 & -I_l \end{pmatrix}.$$

Then we can consider the l-dimensional complex hyperbolic space $\mathbf{H}^l = \mathbf{H}^l(V)$ as an open submanifold of $\mathbf{P}^l = \mathbf{P}(V)$ by

$$\mathbf{H}^l = \{x = (1 : x_1 : \cdots : x_l) \in \mathbf{P}^l \mid |x_1|^2 + \cdots + |x_l|^2 < 1\}.$$

Let W be a Hermitian vector space of dimension $m+1$ and $\mathbf{P}^m = \mathbf{P}(W)$ its projective space. Let $y = (1 : y_1 : \cdots : y_m)$ denote its local affine coordinates with respect to an orthonormal basis of W.

We consider $\mathbf{H}^l$ and $\mathbf{P}^m$ as Kähler manifolds by the well-known Kähler forms

$$- \sqrt{-1} \partial \bar{\partial} \log \left(1 - \sum_i |x^i|^2 \right),$$
$$\sqrt{-1} \partial \bar{\partial} \log \left(1 + \sum_j |y^j|^2 \right),$$

respectively. Then $M = \mathbf{H}^l \times \mathbf{P}^m$ with the product metric is an example of Bochner-Kähler manifolds. Hence the almost complex structure of $Z_k(M)$ is integrable. By the definition of the metric, the group $\mathrm{U}(1, l) \times \mathrm{U}(m + 1)$ acts on M isometrically.

Let $x_0 = (1, 0, \ldots, 0)$ and $y_0 = (1, 0, \ldots, 0)$. By identifying a tangent vector with an element of $x_0^\perp \oplus y_0^\perp$, we can consider it as an element of $V \oplus W$. Let $z^{IJ'}$ be the Plücker coordinate, where I (resp. J) is a multi-index with respect to $x_1, \ldots, x_l$ (resp. $y_1, \ldots, y_m$). Note that the local frame of the tangent bundle that we choose is orthonormal only at (x_0, y_0). Hence (2.1) is valid only at points on the fiber over (x_0, y_0).

Let L be a complex vector space with metric. For a vector $x \in L$, we regard $\bar{x}$ as an element of L^* by the canonical isomorphism $\bar{L} \simeq L^*$. Then, by using the decomposition:

$$\Lambda^{k+1}(V \oplus W) = \bigoplus_{a+b=k+1} \Lambda^a V \otimes \Lambda^b W,$$

we define

$$\phi_{a,b} = x \wedge \left(1 - \frac{y \wedge i(\overline{y})}{(y,y)}\right) z + y \wedge \left(1 - \frac{x \wedge i(\overline{x})}{(x,x)}\right) z \bigg|_{\Lambda^a V \otimes \Lambda^b W}$$

$$= \sum_{|I|=a,|J|=b} \phi_{I,J} e_I \wedge e_{J'},$$

where i denotes the interior multiplication. In the following, we assume that multi-indices $I, J, \ldots$ do not include the index 0. Then we have

(3.1)
$$\phi_{0I,0J} = (-1)^a \left(\sum_{i \notin I} \frac{\overline{x}^i z^{iIJ'}}{(x,x)} + \sum_{j \notin J} \frac{\overline{y}^j z^{j'IJ'}}{(y,y)}\right),$$

$$\phi_{I,0J} = (-1)^a z^{IJ'} - \sum_{i \in I} x^i \phi_{0iI,0J},$$

$$\phi_{0I,J} = z^{IJ'} - \sum_{j \in J} y^j \phi_{0I,0jJ},$$

$$\phi_{I,J} = -\sum_{j \in J} y^j z^{j'IJ'} - \sum_{i \in I} x^i \phi_{0iI,J},$$

where $\bar{x}^i$ is the complex conjugate of x^i.

Let p_1 and p_2 be the projections:

$$p_1 : \mathbf{H}^l \times \mathbf{P}^m \to \mathbf{H}^l,$$

$$p_2 : \mathbf{H}^l \times \mathbf{P}^m \to \mathbf{P}^m.$$

Let $L_{\mathbf{H}^l}$ and $L_{\mathbf{P}^m}$ be the tautological line bundles. Let $L_{Z_k(M)}$ denote the tautological line bundle as a Grassmannian bundle. Then $\phi_{a,b}$ is canonically extended to a map

$$\phi_{a,b} : L_{Z_k(M)} \otimes p_1^*(L_{\mathbf{H}^l}^a) \otimes p_2^*(L_{\mathbf{P}^m}^b) \to \Lambda^a V \otimes \Lambda^b W,$$

which is equivariant under the action of $\mathrm{U}(1,l) \times \mathrm{U}(m+1)$. The space on the left hand side has an almost complex structure induced by that of $Z_k(M)$ and the canonical connection as a line bundle. Then we have the following theorem.

Theorem 3.1. *The map $\phi_{a,b}$ is holomorphic.*

Proof. It suffices to calculate that $\phi_{a,b}$ is $\overline{\partial}$-closed at points on the fiber over (x_0, y_0). Then it is immediate by the definition of the almost complex structure. $\square$

Remark. This theorem shows that the line bundle $L_{Z_k(M)} \otimes p_1^*(L_{\mathbf{H}^l}^a) \otimes p_2^*(L_{\mathbf{P}^m}^b)$ has a natural holomorphic structure if $a + b = k + 1$. In fact, if both l and m are positive, this is necessary, which can be shown directly by computing the integrability condition.

For non-negative integers a and b such that $a + b = k$, put

$$U_{a,b} = \{\kappa \in Z_k(M) \mid \phi_{a+1,b}(\kappa) \neq 0, \phi_{a,b+1}(\kappa) \neq 0\}.$$

These open sets cover $Z_k(M)$. In fact, we have the following lemma.

Lemma 3.2. *Let (x, y, z) be a point of $Z_k(M)$ and L the subspace of $x^\perp \oplus y^\perp$ corresponding to z. Then $(x, y, z) \in U_{a,k-a}$ if and only if $\dim(L \cap x^\perp) \leq a \leq k - \dim(L \cap y^\perp)$.*

Proof. By (3.1), we have

(3.2) $$\phi_{0I,0J}(x_0, y_0, z) = 0,$$

(3.3) $$\phi_{I,0J}(x_0, y_0, z) = (-1)^{|I|} z^{IJ'},$$

(3.4) $$\phi_{0I,J}(x_0, y_0, z) = z^{IJ'},$$

(3.5) $$\phi_{I,J}(x_0, y_0, z) = 0.$$

Hence, by the Plücker relations, we have

$$U_{a,b} = \{(x, y, z) \in Z_k(M) \mid L|_{\Lambda^a x^\perp \otimes \Lambda^b y^\perp} \neq 0\},$$

from which the lemma follows immediately.   □

Now we define a map:

$$\psi_{a,b} : U_{a,b} \to \mathbf{P}(\Lambda^{a+1} V \otimes \Lambda^b W) \times \mathbf{P}(\Lambda^a V \otimes \Lambda^{b+1} W)$$
$$\kappa \mapsto ([\phi_{a+1,b}(\kappa)], [\phi_{a,b+1}(\kappa)]).$$

Theorem 3.3. *The map $\psi_{a,b}$ is an embedding.*

Proof. Since the map is equivariant, to prove the injectivity, it suffices to prove that if $\psi_{a,b}(x, y, w) = \psi_{a,b}(x_0, y_0, z)$ then $x = x_0$, $y = y_0$ and $w = z$. Suppose $(y, y_0) = 0$. Then we can assume $y = (0, 1, 0, \ldots, 0)$. We assume, for a while, that J does not include the index 1. We compute

$$\phi_{0I,0J} = 0 = w^{I0'J'}, \quad |I| = a, a - 1.$$

Hence we have

$$\phi_{I,0J} = (-1)^{|I|} z^{IJ'} = -\sum_{i \in I} x^i \phi_{0iI,0J} = 0, \quad |I| = a,$$

$$\phi_{I,01J} = (-1)^{|I|} z^{I1'J'} = -(-1)^{|I|} w^{I0'J'} = 0, \quad |I| = a.$$

These contradict with the assumption $z \in U_{a,k-a}$.

Hence we should have $(y, y_0) \neq 0$. Then, by (3.3) and (3.4), there exists a constant c such that $w^{IJ'} = cz^{IJ'}$ for all (I, J) with $|I| =$

$a-1, a, a+1$. Hence we have $w = z$ by the Plücker relations. By (3.2) and (3.5), the equations for x and y become

$$\sum_{i \notin I} \frac{\overline{x}^i z^{iIJ'}}{(x,x)} + \sum_{j \notin J} \frac{\overline{y}^j z^{j'IJ'}}{(y,y)} = 0,$$

$$\sum_{j \in J} y^j z^{j'IJ'} + \sum_{i \in I} x^i z^{iIJ'} = 0.$$

Let $x' = (x^1, \ldots, x^l)$ and $y' = (y^1, \ldots, y^m)$. Then these equations mean that there is a k-dimensional subspace L of $x_0^\perp \oplus y_0^\perp$ such that

$$(x', y') \in L \subset (\frac{x'}{(x,x)}, \frac{y'}{(y,y)})^\perp.$$

Hence we have $x = x_0$ and $y = y_0$, concluding that the map is injective.

Injectivity of the derivative is also computed by simple computations at points on the fiber over (x_0, y_0). □

4. THE COMPLEXIFICATION OF THE MODEL SPACES

In this section, we give an explicit description of the *complexification* $M_{\mathbb{C}}$ of the model space $M = \mathbf{H}^l \times \mathbf{P}^m$.

Let $\xi, \zeta \in \mathbf{H}^l$ and $\mu, \nu \in \mathbf{P}^m$ such that $(\mu, \nu) \neq 0$. We construct an embedding of $G_{k,n} = G_k(V/\langle \xi \rangle \oplus W/\langle \mu \rangle)$ in $Z_k(M)$ by generalizing $\phi_{a,b}$ in the previous section. Let a and b be non-negative integers such that $a + b = k + 1$. For $z \in G_k(V/\langle \xi \rangle \oplus W/\langle \mu \rangle)$, we define:

$$\Phi_{a,b}(z) = \xi \wedge \left(1 - \frac{\mu \wedge i(\bar{\nu})}{(\mu, \nu)}\right) z + \mu \wedge \left(1 - \frac{\xi \wedge i(\bar{\zeta})}{(\xi, \zeta)}\right) z \bigg|_{\Lambda^a V \otimes \Lambda^b W}.$$

For non-negative integers a and b satisfying $a + b = k$, put

$$U'_{a,b} = \{z \in G_{k,n} \mid \Phi_{a+1,b}(z) \neq 0, \Phi_{a,b+1}(z) \neq 0\}$$

$$= \{z \in G_{k,n} \mid z|_{\Lambda^a V/\langle \xi \rangle \otimes \Lambda^b W/\langle \mu \rangle} \neq 0\}.$$

Then we define

$$\Psi_{a,b}(\xi, \zeta, \mu, \nu) = \Psi_{a,b} : U'_{a,b} \to \mathbf{P}(\Lambda^{a+1} V \otimes \Lambda^b W) \times \mathbf{P}(\Lambda^a V \otimes \Lambda^{b+1} W)$$

$$z \mapsto ([\Phi_{a+1,b}(z)], [\Phi_{a,b+1}(z)])$$

Theorem 4.1. *Let* $z \in G_{k,n}$. *There exists* $\kappa \in Z_k(M)$ *such that* $\psi_{a,b}(\kappa) = \Psi_{a,b}(z)$ *for all* (a, b) *such that* $z \in U'_{a,b}$. *Hence* $\Psi_{a,b}$ *induces an embedding of* $G_{k,n}$ *in* $Z_k(M)$.

Proof. Let L be the subspace of $V/\langle \xi \rangle \oplus W/\langle \mu \rangle$ corresponding to z. We can assume $\zeta = x_0$ and $\nu = y_0$. By the isomorphisms $x_0^\perp \simeq V/\langle \xi \rangle$ and $y_0^\perp \simeq W/\langle \mu \rangle$ defined by the canonical projections, we can consider

L as a subspace of $x_0^\perp \oplus y_0^\perp$. We give a point $\kappa = (x, y, w)$ which satisfies the property. We fix a and b for a while, and calculate the coefficients of $\Phi_{a+1,b}(z)$ and $\Phi_{a,b+1}(z)$. By a similar argument in the proof of Theorem 3.3, we should have $(y, y_0) \neq 0$. Thus, by (3.1), we have

$$\Phi_{0I,0J}(z) = 0,$$
$$\Phi_{I,0J}(z) = (-1)^{|I|} z^{IJ'},$$
$$\Phi_{0I,J}(z) = z^{IJ'},$$
$$\Phi_{I,J}(z) = -\sum_{j \in J} \mu^j z^{j'IJ'} - \sum_{i \in I} \xi^i z^{iIJ'}.$$

Hence, by the Plücker relations, we have $w = z$. The coordinates of x and y satisfy the equation:

$$(4.1) \qquad \sum_{i \notin I} \frac{\bar{x}^i z^{iIJ'}}{(x, x)} + \sum_{j \notin J} \frac{\bar{y}^j z^{j'IJ'}}{(y, y)} = 0.$$

Define

$$V_1 = L \cap x_0^\perp,$$
$$W_1 = L \cap y_0^\perp.$$

Let V_2 and W_2 be the subspaces such that

$$x_0^\perp = V_1 \oplus V_2$$
$$y_0^\perp = W_1 \oplus W_2$$

which are orthogonal decompositions. Then (4.1) means that

$$x \in \langle x_0 \rangle + V_2,$$
$$y \in \langle y_0 \rangle + W_2.$$

Hence the problem is reduced to the case $V_1 = W_1 = 0$, which implies that $\min(l, m) \geq k$. By considering the canonical anti-holomorphic diffeomorphism:

$$Z_k(M) \to Z_{n-k}(M)$$
$$L \mapsto L^\perp,$$

we can further assume that

$$\dim V = \dim W = k + 1,$$
$$z \in \bigcap_{0 \leq a \leq k} U'_{a,k-a}.$$

In this case, it is easily verified that the equations for x and y are independent with respect to a. Hence it suffices to find x and y when $a = 0$.

An image of the map

$$\Psi_{0,k} : \bigcap_{0 \leq a \leq k} U'_{a,k-a} \to \mathbf{P}(\mathrm{Hom}(W, V))$$

is a projectivized invertible linear map. Hence it suffices to find $x \in V$ and $y \in W$ such that

$$A(y) = x,$$
$$A(y^{\perp}) = x^{\perp},$$

for an arbitrary invertible linear map $A : W \to V$. Indeed, if we find them, we have

$$\psi_{0,k}\left(x, y, \langle A(e_j) + e_j \mid j = 1, \ldots, k \rangle\right) = [A],$$

where $(e_1, \ldots, e_k)$ is a basis of $y^{\perp}$.

We denote the indefinite metric on V as $(,)_G$. By the automorphism A, we define a Hermitian metric $(,)_H$ on V. For a positive number r, let S_r be the real hypersurface of V defined as

$$S_r = \{v \in V \mid (v, v)_H = r\}.$$

Let H be the real hypersurface of V defined as

$$H = \{v \in V \mid (v, v)_G = 1\}.$$

Let r_0 be the minimum number such that $S_r \cap H \neq \emptyset$. Then we can show easily that S_{r_0} and H intersect at one point x. Then x and $y = A^{-1}(x)$ satisfy the desired property. □

Let (ξ, ζ, μ, ν) be as above. We denote the corresponding submanifold of $Z_k(M)$ as $Z_k(\xi, \zeta, \mu, \nu)$. The intersection of two such submanifolds can be completely determined as follows.

Theorem 4.2. *Two submanifolds $Z_k(\xi_i, \zeta_i, \mu_i, \nu_i)$, $i = 1, 2$ intersect if and only if*

$$\frac{(\xi_1, \zeta_2)(\xi_2, \zeta_1)}{(\xi_1, \zeta_1)(\xi_2, \zeta_2)} = \frac{(\mu_1, \nu_2)(\mu_2, \nu_1)}{(\mu_1, \nu_1)(\mu_2, \nu_2)}.$$

When this condition is satisfied, the intersection is one of the following:

$$\begin{aligned}
G_{k,n-1}, & \quad \text{if } \xi_1 = \xi_2, \ \mu_1 = \mu_2, \\
G_{k-1,n-1}, & \quad \text{if } \zeta_1 = \zeta_2, \ \nu_1 = \nu_2, \\
G_{k-1,n-2}, & \quad \text{otherwise.}
\end{aligned}$$

Proof. We can put $\zeta_2 = x_0$ and $\nu_2 = y_0$. By a similar argument in the proof of Theorem 3.3, we can show that two submanifolds are disjoint if $(\mu_1, y_0) = 0$. Hence we can assume that $(\mu_1, y_0) \neq 0$. Let z and w be elements of $G_k(x_0^\perp \oplus y_0^\perp)$ such that $\Psi_{a,b}(\xi_1, \zeta_1, \mu_1, \nu_1)(z) = \Psi_{a,b}(\xi_2, x_0, \mu_2, y_0)(w)$. Then, since $\Phi_{0I,0J}(z) = 0$ for $|I| = a, a-1$, we have $z = w$. Let $\xi_1 = (1, \xi_1^1, \dots, \xi_1^l)$ and $\mu_1 = (1, \mu_1^1, \dots, \mu_1^m)$. Then the equations are written as

$$
(4.2) \qquad
\begin{aligned}
\frac{\sum_{i \notin I} \bar{\zeta}_1^i z^{iIJ'}}{\bar{\zeta}_1^0 - \sum_i \xi_1^i \bar{\zeta}_1^i} + \frac{\sum_{j \notin J} \bar{\nu}_1^j z^{j'IJ'}}{\bar{\nu}_1^0 + \sum_j \mu_1^j \bar{\nu}_1^j} &= 0, \\
\sum_{i \in I} (\xi_1^i - \xi_2^i) z^{iIJ'} + \sum_{j \in J} (\mu_1^j - \mu_2^j) z^{j'IJ'} &= 0.
\end{aligned}
$$

Let L be the subspace of $x_0^\perp \oplus y_0^\perp$ corresponding to z and $\xi_i' = (0, \xi_i^1, \dots, \xi_i^l)$, $\mu_i' = (0, \mu_i^1, \dots, \mu_i^m)$, $i = 1, 2$. Then (4.2) means that

$$
(\xi_1' - \xi_2', \mu_1' - \mu_2') \in L \subset \left(\frac{\zeta_1}{(\zeta_1, \xi_1)}, \frac{\nu_1}{(\nu_1, \mu_1)} \right)^\perp.
$$

Then the theorem follows immediately. $\qquad\square$

Corollary 4.3. *Let $M = \mathbf{H}^l \times \mathbf{P}^m$. Then we have*
$$
M_{\mathbb{C}} = \mathbf{H}^l \times \bar{\mathbf{H}}^l \times \left(\mathbf{P}^m \times \bar{\mathbf{P}}^m \setminus F(1, m; m+1) \right).
$$

5. Some examples

1. $M = \mathbf{P}(W) = \mathbf{P}^n$: We can write $Z_k(\mathbf{P}^n) = F(k, k+1; n+1)$. The projection map is written as
$$
F(k, k+1; n+1) \to \mathbf{P}^n
$$
$$
(U, U') \mapsto U^\perp \cap U'
$$
The complexification $M_{\mathbb{C}}$ is $\mathbf{P}^n \times \bar{\mathbf{P}}^n \setminus F(1, n; n+1)$. For $(\mu, \nu) \in M_{\mathbb{C}}$, the corresponding submanifold is written as
$$
\{(U, U') \mid U \subset \nu, \mu \subset U'\}.
$$
This is a straightforward generalization of the well-known description of the twistor space of $\mathbf{P}^2$.

2. $M = \mathbf{H}(V) = \mathbf{H}^n$: For a subspace U of V, we write
$$
U = \begin{cases} U_\pm, & \text{if } U \text{ has a vector with positive norm,} \\ U_-, & \text{if } G|_U \text{ is negative definite.} \end{cases}
$$
Then we have
$$
Z_k(\mathbf{H}^n) = \{(U_-, U_\pm') \in F(k, k+1; V)\}
$$

The projection map is written as:

$$Z_k(\mathbf{H}^n) \to \mathbf{H}^n$$
$$(U, U') \mapsto U^\perp \cap U'$$

$M_{\mathbb{C}}$ is $\mathbf{H}^n \times \bar{\mathbf{H}}^n$. For $(\xi, \zeta) \in M_{\mathbb{C}}$, the corresponding submanifold is written as:

$$\{(U, U') \mid U \subset \zeta, \xi \subset U'\}$$

3. $Z_1(\mathbf{H}^l \times \mathbf{P}^m)$ with $l, m \geq 1$: The restriction of $\phi_{1,1}$ to $U_{1,0} \cap U_{0,1}$ is an isomorphism to the projectivization of the space of rank 2 elements of $V \otimes W$. The boundary of the image is identified with $\mathbf{P}(V) \times \mathbf{P}(W)$ by the Segre embedding. The space $\mathbf{P}(V)$ has a decomposition:

$$\mathbf{P}(V) = \mathbf{H}_+ \cup S^{2l-1} \cup \mathbf{H}_-,$$

by the sign of the norm. Then we obtain $Z_1(\mathbf{H}^l \times \mathbf{P}^m)$ by changing the boundary as follows:

(a) remove $S^{2l-1} \times \mathbf{P}^m$,

(b) replace $\mathbf{H}_+ \times \mathbf{P}^m$ with $\mathbf{P}(T^{(1,0)}\mathbf{H}_+) \times \mathbf{P}^m$,

(c) replace $\mathbf{H}_- \times \mathbf{P}^m$ with $\mathbf{H}_- \times \mathbf{P}(T^{(1,0)}\mathbf{P}^m)$.

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